Correlation Structure between Inflation and Oil Futures Returns: An Equilibrium Approach

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Abstract

We use an equilibrium model of a monetary economy to understand the economics behind the correlation between inflation and oil futures returns. We find that some of the positive correlation found in empirical studies is due to the fact that oil is in the consumption basket, however, this accounts only for a minor part of it. There exist other important sources of correlation related to monetary shocks and output shocks. In particular, we find that the correlation is extremely sensitive to the reaction of the central bank to output shocks, while the reaction to inflation changes is less significant. We estimate our model using maximum likelihood with the following datasets: crude oil futures prices, nominal interest rates, inflation rates and money supply growth rates. Our estimates suggest that the monetary authority overreacts to output shocks by increasing the money supply in a more than necessary amount, generating a significant source of positive correlation. From a practical perspective, We find that it is a good strategy to use as a hedge, the futures whose maturity is closer to the hedging horizon. This is particularly true for short-term hedging.

1 Introduction

In a monetary economy, producers and workers receive income in money form; however, their utility depends on the goods the money allows them to purchase. As a result, even if they can guarantee a certain amount of money in the future, these agents are still exposed to inflation risk. Although these economies typically have central banks which serve to stabilize and set target levels (or ranges) for the inflation rate, such banks are generally unable to completely eliminate the price level volatility. Therefore, arises the need to hedge against this type of risk.

A necessary condition for hedging against inflation risk is having access to assets whose returns are strongly correlated (positively or negatively) with the inflation rate. Several studies have reported a positive and significant correlation, both statistically and economically, between inflation and commodity futures returns. For example, Bodie and Rosansky (1980), use a sample for the 1950-1976 period and find a correlation of 0.58 between the nominal annual returns of a portfolio of commodity futures and the inflation rate. Gorton and Rouwenhorst (2006) report somewhat lower values for this correlation using a sample period between 1959 and 2004, but demonstrate that such correlations increase with the investment horizon. Furthermore, using quarterly returns they find that the correlation of commodities with unexpected inflation is 0.25, almost double the correlation with total inflation, which is 0.14 in their sample. Similarly, based on annual data between 1970 and 1999, Greer (2000) reports correlation values of 0.59 and 0.23, respectively.

Our paper contributes by offering an economic explanation for the correlation between inflation and commodity futures returns. As far as we know, this is the first paper that goes beyond intuition to understand the origin of this relation. We concentrate on oil futures as the hedging instruments, because commodities with a high energy component have been indicated to be the most powerful financial hedges against inflation (see e.g. Froot 1995). We use a production economy with a monetary sector to back out equilibrium prices and futures prices. This allows us to study the joint dynamics of inflation and oil futures returns and, in particular, the correlation structure between these two variables.

Money is incorporated on the demand side through real monetary holdings in the utility function, as in Sidrauski (1967), Bakshi and Chen (1996), Buraschi and Jiltsov (2005) and Lioui and Poncet (2005).[1] Following the idea of the simplest Taylor's rule (see Taylor 1993), we assume that the money supply process is driven by a central bank with real and monetary targets. In our economy, the role of oil is twofold. First, units of oil can be consumed by the agents. These units represent the fuel consumed for heating and transportation that is included in the basket used to calculate the CPI. Second, oil is explicitly modeled as an input for the production of

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[1] Another approach widely used to incorporate money in the economy is the Cash-in-Advance mechanism of Lucas and Stokey (1987), and extended by Balduzzi (2007) to a continuous-time framework.

capital, therefore, it affects output and generates a second mechanism that relates oil to the money supply process.

The correlation between inflation and commodity futures returns has two temporal dimensions, one related to the hedging horizon and the other to the maturity of the contract used as the hedging instrument. Our empirical results show that for periods shorter than one year, a futures whose maturity equals to the holding period is better as an inflation hedge. For longer hedging horizons, differences in correlation across maturities are not significant. These results provide support for the strategy used in studies like Bodie and Rosansky (1980), Gorton and Rouwenhorst (2006) and Herbst (1985). Moreover, our results agree with Gorton and Rouwenhorst (2006), who find that the correlation increases with the holding period regardless of the maturity used.

To better understand the properties of the model, we consider two interesting limiting cases for the correlation: (i) when the holding period tends to zero (i.e. an infinitesimal hedging horizon) and, (ii) when both the holding period and the maturity of the futures contract tend to zero. The first case studies the instantaneous correlation between inflation and futures returns for different maturities. The second case represents the instantaneous correlation between inflation and oil spot returns and provides a clear explanation of the positive correlation found in the literature. We find that monetary and oil stock shocks are natural sources of the positive correlation. Output shocks originally cause a negative correlation, however, a strong reaction of the monetary authority to this type of shocks may reverse this effect. Indeed, we find that the endogenous monetary policy reaction to output shocks are crucial for the correlation between inflation and future returns, while the reaction to inflationary shocks have only a minor effect in this correlation. We decompose the equilibrium covariance between inflation and oil spot returns in three components associated to the exogenous shocks considered in our model. Our estimates imply that 88.7% of this covariance is due to the output shocks, 0.5% comes from the oil stock shocks and the remaining 10.8% is associated to the monetary shocks.

Our paper also contributes to the current commodity literature by correctly treating futures prices as nominal variables. Recent papers that have studied commodity prices using equilibrium models in real terms (see for example, Carlson, Khokher, and Titman 2007, Casassus, Collin-Dufresne, and Routledge 2008, and Kogan, Livdan, and Yaron 2009), have used nominal futures prices to perform their estimation without specifying any price adjustment due to inflation. Ignoring the inflationary component contained in futures returns increases the potential of biasing the estimates of these models. Also, reduced form models of valuation and hedging with commodity futures have treated inflation with ambiguity. Many papers use a mean-reverting process for the nominal spot price dynamics, a specification which fits better real variables (see for example, Casassus and Collin-Dufresne 2005, and Model 1 of Schwartz 1997). If we interpret their spot price process in real terms, they would fall into the inconsistency of calculating the process parameters using nominal futures prices.

Finally, there is an open macroeconomic debate about the role of the monetary policy that responded to oil-price shocks in causing US recessions (see Bernanke, Gertler, and Watson 1997, Bernanke, Gertler, and Watson 2004, and Hamilton and Herrera 2004). We differ from this literature in that in our model the oil price and the monetary policy are both endogenous. We do not explicitly allow for a reaction to oil-price shocks, instead, the monetary authority reacts to variables that affect oil prices.

The rest structure of the paper is as follows. The following section presents the general equilibrium model and its basic implications. Section 3 contains the empirical methodology and aspects related to the estimation of model. Section 4 discusses the correlation structure between inflation and oil futures returns. Section 5 concludes. The proofs of all propositions are in the appendices.

2 An equilibrium model of oil and inflation

This section proposes a model based on a two-sector two-good monetary economy. We present the basic ingredients of the model, propose an approximate solution to the social planner's problem, and present the main findings useful to understand the correlation between inflation and futures returns: the optimal consumption/input strategy, real oil price, the equilibrium inflation rate and the valuation of oil futures contracts. The real side of the model follows Casassus, Collin-Dufresne, and Routledge (2008) with some simplifying assumptions necessary to get closed-form expressions.

2.1 The model

The representative agent personifies in proportions ϕ and $1-\phi$ two types of individuals. Agents that belong to the first category are firm owners who have direct access to a consumption good and whose consumption is C_t. The second type of individuals is composed by agents who do not have access to goods and require money to carry out consumer transactions. This motivates the real monetary holdings in the utility function, $m_t \equiv \frac{M_t^d}{P_t}$, where M_t^d is the money demand and P_t is the price level.[2] The utility function of the representative agent is assumed to be

$$U(C_t, m_t) = \phi \log(C_t) + (1 - \phi) \log(m_t) \tag{1}$$

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[2]Our paper considers that the only purpose of money is for consumer transactions, therefore, money is not preserved and does not generate a loss due to inflation. A central bank supplies money at no cost and provides it as a transfer to the second type of individuals. Although the government sector is not explicitly modeled in our economy, money can be considered as a debt that does not generate interest and which is subsequently paid by the government through lump-sum taxation of company owners.

The second type of agents uses money to finance a consumption basket, X_t, which implies that $m_t = X_t$. The basket is a combination of the consumption good and units of oil, denoted as Z_t and Y_t, respectively, with a share of the consumption good of α. For tractability, we assume that X_t has the following structure

$$X_t = Z_t^\alpha Y_t^{1-\alpha} \tag{2}$$

Crude oil lies underground in infinite quantities, however, extraction requires certain investments. The oil stocks for direct consumption or production, Q_t, is assumed as follows

$$dQ_t = (-\gamma_t - Y_t + \psi\, i_t Q_t)dt + \sigma_Q Q_t dW_{Qt} \tag{3}$$

where γ_t is the quantity of oil used as an input for production of capital, $\psi i_t Q_t$ are the new units of oil extracted from the ground, ψ is an investment productivity parameter, σ_Q is the volatility of the diffusion and W_{Qt} is a standard Brownian motion.

The economy produces a capital good using capital and oil through the following technology

$$f(K_t, \gamma_t) = A\, K_t^{1-\eta}\gamma_t^\eta \tag{4}$$

where A is the total factor productivity and K_t is the capital stock. Capital can be invested to produce more capital, consumed by the agents (directly or through the consumption basket), or invested to extract new units of oil. Therefore, the dynamics of the capital stock is

$$dK_t = (f(K_t, \gamma_t) - C_t - Z_t - i_t K_t)dt + \sigma_K K_t dW_{Kt} \tag{5}$$

where $i_t K_t$ is the investment in new units of oil, σ_K is the volatility of the process and W_{Kt} is a standard Brownian motion which is assumed to be independent from W_{Qt}. The stochastic component $\sigma_K K_t dW_{Kt}$ corresponds to the output shocks.

As it will become clear later, the investment process is important to match the variability in the slope of the futures curve. Indeed, it can be shown that our model predicts that the main source of this variability, the convenience yield, is solely a function of the investment rate, i_t.[3] We will also see that the investment rate does not affect prices nor the correlation between inflation and oil spot returns. Considering that the main purpose of the investment process is fitting the data, we assume an ad-hoc exogenous specification that allows us to obtain a simple solution of the model instead of endogenizing the investment decision. Therefore, we assume that the investment rate is driven by the following process

$$di_t = \kappa_i(\theta_i - i_t)dt + \sigma_i dW_{it} \tag{6}$$

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[3]The convenience yield is defined as the implied benefit associated with holding the underlying physical good, in this case, a barrel of oil. Empirical studies such as Schwartz (1997) or Casassus and Collin-Dufresne (2005) suggest that the variability of slope in the futures curve is mostly explained by changes in the convenience yield, rather than by changes in interest rates.

where κ_i is the speed of adjustment parameter, θ_i is the long-term investment rate, σ_i is the volatility and W_{it} is a the standard Brownian motion independent from W_{Kt} and W_{Qt}.

The price level in the economy, P_t, is endogenous and completely flexible to balance the monetary market. Thus, given a level of money supply, M_t, the price level adjusts instantaneously so that the optimally chosen level for consumption basket X_t, is affordable. There is a central bank that implements the monetary policy, however, it does not perfectly control the growth of M_t. The following process is assumed for the money supply

$$\frac{dM_t}{M_t} = \mu_M dt + b_K \left(\frac{dK_t}{K_t} - \theta_K dt \right) + b_P \left(\frac{dP_t}{P_t} - \theta_P dt \right) + \sigma_M dW_{Mt} \tag{7}$$

The money supply process has an exogenous growth component (μ_M) and two endogenous components that react to deviations from output growth and inflation targets, denoted by θ_K and θ_P, respectively. This specification resembles a standard Taylor's rule that relates the monetary policy instrument, the nominal fed funds interest rate, to the output gap and deviations from the target inflation rate.

The coefficients b_K and b_P measure the speed-of-adjustment to the target levels θ_K and θ_P. If $b_K > 0$ and output is above (below) the target level θ_K, the central bank increases (decreases) the money supply. If $b_P < 0$ and the inflation rate is above (below) the target level θ_P, the central bank reduces (increases) the money supply. The coefficient σ_M is the constant volatility of the process and W_{Mt} is a Brownian motion independent of those previously defined.

2.2 Social planner equilibrium

In the absence of market frictions, the first welfare theorem guarantees that the decentralized equilibrium, or equilibrium through the price system, is equivalent to that of a social planner. Thus, it is assumed that there is a social planner that solves the following problem,

$$\mathcal{J}(K_0, Q_0, i_0, 0) \equiv \sup_{\{C_t, Z_t, Y_t, \gamma_t\} \in \Psi} \mathbb{E}_0 \left[\int_0^\infty e^{-\beta t} U(C_t, m_t) dt \right] \tag{8}$$

subject to

$$dK_t = (f(K_t, \gamma_t) - C_t - Z_t - i_t K_t) dt + \sigma_K K_t dW_{Kt} \tag{9}$$

$$dQ_t = (-\gamma_t - Y_t + \psi\, i_t Q_t) dt + \sigma_Q Q_t dW_{Qt} \tag{10}$$

$$di_t = \kappa_i (\theta_i - i_t) dt + \sigma_i dW_{it} \tag{11}$$

where $\mathcal{J}(K_t, Q_t, i_t, t)$ is the value function of the problem and β is the subjective discount factor of the representative agent. Equations (8)-(11) do not include the money supply dynamics as a restriction. The justification for this is monetary neutrality; that is, because the price level is fully flexible, money does not affect the real economy. Indeed, M_t is not a state variable for the dynamic programming problem in the real economy.

Since $\mathcal{J}(K_t, Q_t, i_t, t)$ depends on t only through the discount factor, it is possible to define the current value function $J(K_t, Q_t, i_t)$, such that, $\mathcal{J}(K_t, Q_t, i_t, t) = e^{-\beta t} J(K_t, Q_t, i_t)$. The current value function is independent of t and satisfies the following Hamilton-Jacobi-Bellman (HJB) equation,

$$0 = \max_{\{C_t, Z_t, Y_t, \gamma_t\}} \left\{ U(C_t, m_t) + \frac{\mathbb{E}_t[dJ]}{dt} - \beta J \right\} \tag{12}$$

subject to

$$\lim_{t \to \infty} \mathbb{E}_0 \left[e^{-\beta t} J(K_t, Q_t, i_t) \right] = 0$$

We express real quantities in terms of consumption basket units (i.e., we choose the consumption basket as the numeraire) and obtain the real oil price by utility indifference. The oil price corresponds to the number of consumption basket units the representative agent is willing to give for an extra unit of commodity (i.e. the shadow price). The equilibrium real oil price is as follows:

$$S_{rt} = \frac{J_Q}{J_K} X_K \tag{13}$$

where J_j and X_j denote the first order partial derivatives of $J(K_t, Q_t, i_t)$ and $X(K_t, Q_t)$ with respect to variable j, respectively.[4] To interpret equation (13), we note that the relative price $\frac{J_Q}{J_K}$ represents the marginal valuation of a unit of oil in terms of units of capital, while the factor X_K converts units of capital to units of consumption basket.

2.3 Approximate solution to the social planner's problem

Due to the functional form chosen for the production function, the problem in equation (12) does not have an exact analytical solution. A simple way to approach this difficulty is by using an approximation of $f(K_t, \gamma_t)$ as a substitute for it.[5] Here, we consider a first-order Taylor approximation of the production function around $\eta = 0$. Recent real business cycle (RBC) studies that include energy as a production factor have used $\eta = 0.04$ for the oil share of income, which justifies the expansion of this variable around $\eta = 0$ (see Finn 2000, and Wei 2003). Therefore, the approximated production function is

$$f(K_t, \gamma_t) = A K_t (1 + \eta(\log(\gamma_t) - \log(K_t))) \tag{14}$$

The following proposition summarizes the optimal consumption and input decisions when using the production function in equation (14).

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[4]To see this result define $H(X(K_t, Q_t), Q_t, i_t) \equiv J(K_t, Q_t, i_t)$. In equilibrium, S_{rt} must be such that the individual is indifferent from demanding an infinitesimal extra quantity ε of oil, that is, $H(X(K_t, Q_t), Q_t, i_t) = H(X(K_t, Q_t + \varepsilon) + S_{rt}\varepsilon, Q_t - \varepsilon, i_t)$. A first-order Taylor expansion to this indifference condition around $\varepsilon = 0$, yields $S_{rt} = \frac{H_Q}{H_X} + X_Q$. Expressing these partial derivatives in terms of the value function J and its derivatives becomes equation (13).

[5]An alternative approach would be to use a Taylor expansion for the value function (see for example, Kogan 2001, and Janecek and Shreve 2004).

Proposition 1 *The current value function $J(K_t, Q_t, i_t)$ has an affine solution in $\log(K_t)$, $\log(Q_t)$ and i_t. The optimal consumption of capital for each type of individual are*

$$C_t^* = \frac{\beta\phi}{1 - \beta\delta_\omega} K_t \quad and \quad Z_t^* = \frac{\alpha\beta(1 - \phi)}{1 - \beta\delta_\omega} K_t \tag{15}$$

The optimal quantities of oil used for consumption and production are

$$Y_t^* = \frac{(1 - \phi)(1 - \alpha)}{\delta_\omega} Q_t \quad and \quad \gamma_t^* = A\eta \frac{1 - \beta\delta_\omega}{\beta\delta_\omega} Q_t \tag{16}$$

Finally, the real price of oil is given by

$$S_{rt} = \frac{\alpha^{1+\alpha}\beta^{1+\alpha}\delta_\omega^\alpha(1 - \alpha)^{1-\alpha}(1 - \phi)}{(1 - \beta\delta_\omega)^{1+\alpha}} e^{-\alpha\omega_t} \tag{17}$$

Here, δ_ω is a parameter of the solution of the value function and ω_t is the logarithm of the ratio of oil stocks to capital stocks, i.e., $\omega_t = \log\left(\frac{Q_t}{K_t}\right)$.

Proof See Appendix A.1. □

2.4 Inflation rate and oil futures returns

In this section we obtain the equilibrium value for the two main variables in our study: the inflation rate and futures prices. First, recall that agents use money only to finance the consumption basket, therefore,

$$M_t^d = P_t X_t = P_t Z_t^\alpha Y_t^{1-\alpha} = P_t K_t^\alpha Q_t^{1-\alpha} \left(\frac{\alpha\beta(1 - \phi)}{1 - \beta\delta_\omega}\right)^\alpha \left(\frac{(1 - \phi)(1 - \alpha)}{\delta_\omega}\right)^{1-\alpha} \tag{18}$$

Applying Itô's lemma to equation (18) yields an expression for the money demand dynamics. The economy's inflation rate can be determined using the monetary market clearing condition, $M_t^d = M_t$. This condition is satisfied at any time, hence, the dynamics of both, money demand and money supply, must be identical. The price process P_t adjusts to make these dynamics equivalent. The following proposition presents the price process that clears the money market.

Proposition 2 *The equilibrium price process is given by the following expression:*

$$\frac{dP_t}{P_t} = \left(\mu_{P0} + \frac{A\eta(b_K - \alpha)}{1 - b_P}\omega_t - \frac{\psi(1 - \alpha) + b_K - \alpha}{1 - b_P}i_t\right)dt + \sigma_{PK}dW_{Kt} + \sigma_{PQ}dW_{Qt} + \sigma_{PM}dW_{Mt} \tag{19}$$

where the constant term on the drift, μ_{P0}, is function of the structural parameters shown in the appendix and

$$\sigma_{PK} \equiv \frac{\sigma_K(b_K - \alpha)}{1 - b_P}, \qquad \sigma_{PQ} \equiv -\frac{\sigma_Q(1 - \alpha)}{1 - b_P} \quad and \quad \sigma_{PM} \equiv \frac{\sigma_M}{1 - b_P}. \tag{20}$$

Proof See Appendix A.2. □

The inflation rate is the change in prices, $\frac{dP_t}{P_t}$. Therefore, the expected inflation rate is $\mathbb{E}_t\left[\frac{dP_t}{P_t}\right]$ and the conditional variance is $\mathbb{V}ar_t\left[\frac{dP_t}{P_t}\right]$.

The crude oil futures contracts can be freely traded in nominal terms for any maturity. We define $\tau = T - t$ as the maturity of a contract the expires at time T. Since these contracts do not generate cash flows when they are signed, the futures price is a martingale under the risk-neutral measure $\mathbb{Q}$, i.e.,

$$\mathbb{E}_t^{\mathbb{Q}}\left[dF(\omega_t, i_t, p_t, \tau)\right] = 0 \tag{21}$$

where $p_t \equiv \log P_t$. Furthermore, the following terminal condition must be satisfied to guarantee the absence of arbitrage

$$F(\omega_T, i_T, p_T, 0) = S_T \equiv P_T S_{rT} \tag{22}$$

where S_T is the nominal oil spot price at the time when the contract expires. The following proposition shows that the futures prices can be easily obtained in our model.

Proposition 3 *Futures prices are exponentially affine functions of ω_t, i_t and p_t,*

$$F_t = F(\omega_t, i_t, p_t, \tau) = \exp(B_{F0}(\tau) + B_{F\omega}(\tau)\omega_t + B_{Fi}(\tau)i_t + p_t) \tag{23}$$

where the $B_{Fj}(\tau)$, $j \in \{0, \omega, i, p\}$, are deterministic functions of the maturity τ defined in the appendix.

Proof See Appendix A.3. □

Our model is sufficiently flexible and general. Indeed, it has been contextualized for oil, but is adaptable to any productive commodity. A common disadvantage of general equilibrium models are the complexity and number of parameters. However, this not the case for our model, because our simplifying assumptions allow us to obtain closed-form expressions for the variables of interests. Indeed, our model is comparable to a three-factor affine reduced form model for futures prices, price level and bond prices, but has the advantage of being built upon equilibrium. Our model enhances the economics behind the parameters. The model described above can be easily extended to consider other sources of uncertainty, as long as the affine nature of the model is kept.

3 Empirical implementation

In this section we first describe the data and the empirical methodology. Then, we discuss the estimates and the in-sample fit of the model.

We use four data sets for the estimation of the model: crude oil futures prices, nominal interest rates, inflation rates and money supply growth rates. We have 210 monthly observations from July 1992 to December 2009 for each one of the variables in the data sets. Crude oil futures are from The New York Mercantile Exchange (NYMEX). End-of-month futures for 8 different contracts with maturities ranging from 1 to 30 months are considered. These maturities are enough to capture the variability in short-term futures prices and in the slope of the futures curve. The maturity of the contracts varies over time, but this is not an issue when monthly observations are used because there is always a price for the each maturity. For the nominal interest rates, the 3-month constant-maturity Treasury yields is used. Inflation data is based on the seasonally adjusted Consumer Price Index (CPI) for all urban consumers from the U.S. Bureau of Labor Statistics. We follow Buraschi and Jiltsov (2005) and build the money supply growth rate from the seasonally adjusted M2 money stock measure obtained from The Federal Reserve Board of Governors. The "Historical data" columns in Table 1 describe the summary statistics for the observed variables.

The model is estimated by maximum likelihood using both time-series and cross sectional data. The model in real terms presented in equations (8)-(11) is stationary because it depends on two stationary variables: the (log) commodity-capital stock ratio, ω_t, and the investment rate, i_t.[6] For the monetary economy, the (log) price level, p_t, which follows a non-stationary process, is also consider. This generates some inconsistencies with the standard maximum likelihood methodology that assumes stationarity for the state variables. To overcome this inconvenience, we note that among the observed variables, only futures prices depend on the non-stationary variable p_t. Using equation (23), we create a new variable that only depends on stationary variables

$$z(\omega_t, i_t, \tau) = \log(F(\omega_t, i_t, p_t, \tau)) - p_t = B_{F0}(\tau) + B_{F\omega}(\tau)\omega_t + B_{Fi}(\tau)i_t \qquad (24)$$

The adjusted futures $z(\omega_t, i_t, \tau)$ are different from the futures prices in a real world, because there exist correlation between the price level and the futures prices.[7] To check this, note that the solutions of $B_{F0}(\tau)$, $B_{F\omega}(\tau)$ and $B_{Fi}(\tau)$ in Appendix A.3 depends on the parameters of the price process P_t, while the futures prices in a real economy should be independent of these parameters.

To link the observed data with the latent variables, we consider 4 measurement, or pricing, equations: the adjusted futures, $z(\omega_t, i_t, \tau)$; three-month nominal yields; the expected inflation rate, $\mathbb{E}_t[dP_t/P_t]/dt$; and the expected money supply growth rate, $\mathbb{E}_t[dM_t/M_t]/dt$. All pricing equations are affine in the two stationary variables, ω_t and i_t. Following Chen and Scott (1993) and Pearson and Sun (1994), we arbitrary choose two observations to pin down the two state variables. In particular, we assume that the 6- and 30-month adjusted futures are observed with

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[6]To see that ω_t follows a stationary process, apply Itô's lemma to $\omega_t = \log(Q_t/K_t)$.

[7]A futures contract in a real world is a contract that pays a unit of real crude oil at maturity.

no error. The remaining cross-sectional data are over-identified, therefore, we assume that they are observed with "measurement errors" which follow AR(1) processes. For simplicity it is assumed that the measurement errors in the adjusted futures prices have the same autocorrelation coefficient for all maturities.

Given the Gaussian nature of our model, the exact likelihood function can be derived in closed form. We leave some free parameters to be estimated, while we calibrate others. In particular, the parameters σ_K, A, κ_i, θ_i, σ_i, μ_M, b_K and b_P are estimated, while β, ϕ, α, η, ψ, σ_Q, θ_K, θ_P and σ_M are calibrated. First, the calibrated parameters denoted with an asterisk in Table 2 are discussed. The subjective discount factor is assumed to be $\beta = 0.05$, which is a standard value in the RBC literature. The preference parameter $\phi = 0.687$ is obtained from Balduzzi (2007) which is also close to the value used in Buraschi and Jiltsov (2005). The non-energy share in the consumption basket, $\alpha = 0.912$, is obtained from the relative importance of non-energy components in the Consumer Price Indexes (U.S. city average) reported by the U.S. Bureau of Labor Statistics for December 2009. As mentioned before, the oil share of income is $\eta = 0.04$, which is in line with recent RBC studies that include energy as a production factor (e.g. Finn 2000, Wei 2003, and Casassus, Collin-Dufresne, and Routledge 2008). The productivity parameter ψ, is calibrated using the worldwide average lifting costs reported by oil and natural gas production companies to the U.S. Energy Information Administration. In the data, the average production costs for 2008 was 8.54 US$/bbl, while in our model, the average cost in nominal terms is $P_t S_{Kt} i_t K_t / (\psi i_t Q_t)$ with S_{Kt} being the price of one unit of capital. Using the unconditional mean of ω_t and the price level for 2008 yields $\psi = 52.30$. For the volatility of the oil stock returns, we calculate the standard deviation of annual changes of worldwide oil consumption from the U.S. Energy Information Administration. This results in $\sigma_Q = 0.015$.[8] We assume that the target parameters, θ_K and θ_P, are equivalent to the unconditional mean of the capital growth rate and the inflation rate in our model. Therefore, we use $\theta_K = \mathbb{E}_t[dK_t/K_t]/dt = 0.037$ and $\theta_P = \mathbb{E}_t[dP_t/P_t]/dt = 0.026$. Note that these parameters need to calibrated, because they are not identified from the data. Indeed, with the data sets used for the ML estimation, only the term $\mu_M - b_K \theta_K - b_P \theta_P$ can be identified from the money supply process. Finally, volatility of the money growth rate is calculated directly from the data, i.e. $\sigma_M = 0.024$.

Table 2 shows the maximum likelihood estimates and significance of the parameters. The volatility of the capital growth rate is $\sigma_K = 0.333$. This figure implies that consumption in the model is more volatile than in the data, which is a well-known failure of RBC models when they are estimated with financial data. The total productivity factor is $A = 0.143$ which is reasonable considering that the capital share of output is $1 - \eta = 0.96$. We obtain that the long-term investment rate is a low fraction of capital, $\theta_i = 0.001$, and that the investment rate shocks have a half-life close to one year (i.e. $\log(2/\kappa_i) = 1.13$) and volatility $\sigma_i = 0.003$. The

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[8]In the model, consumption is a constant fraction of the oil stock; therefore, both have the same growth rate or return volatility.

money supply growth rate parameter is $\mu_M = 0.056$ which is close the sample average growth rate 0.054 (see Table 1). The reaction of the monetary policy to output is $b_K = 0.946 > 0$, which means that the monetary authority injects money when output is above the threshold. A positive value of b_K close to α makes sense because it decreases the volatility of the price process with respect to output shocks (see equation (19)). Conversely, the reaction of the monetary policy to inflation is negative and significant ($b_P = -0.391$) which suggests that the authority fights inflation. Table 2 also reports the estimates for the autocorrelation of the measurement errors for the adjusted futures prices, 3-month yields, expected inflation and expected money supply growth rate, denoted as ρ_1, ρ_2, ρ_3 and ρ_4, respectively.

Finally, Table 1 shows the in-sample fit of the model for the different data sets. The futures prices are calculated from the implied adjusted futures prices and the price level. The table shows that the 6- and 30-month fit perfectly by construction. The other implied futures prices have similar first and second moments to those in the data. What is more surprising is the good is-sample fit of the data that is not used to back out the latent state variables. The average of the implied 3-month yields, inflation rate and money supply growth rate are very close to their corresponding values in the data.

4   Correlation between inflation and oil futures returns

The purpose of this section is to show an expression and numerical values for the conditional correlation between inflation and oil futures returns, based on the economic foundations contained in the proposed model.

It is important to consider, for the correct analysis and interpretation of the results, that the correlation between inflation and future returns has two temporal dimensions: the holding period (hedging horizon) and the maturity of futures contract used as a hedging instrument. Agents might want to hedge for different horizons and using futures contracts with different maturities. If we fix the maturity of a particular futures contract and vary the hedging horizon, we can see how good are the hedging characteristics of that particular instrument for different holding periods.[9] On the other hand, if we fix the hedging horizon, there is a whole bunch of futures contract that can be used to hedge the inflation risk. The following proposition formally presents these results.

Proposition 4 *The conditional correlation between the inflation rate and the futures returns, considering a holding period Δ and that the futures matures τ periods from now (with $\tau \geq \Delta$),*

---

[9]It is important to highlight that nothing prevents from using futures contracts with maturities less than the holding period. However, the analysis here is restricted to maturities larger than the hedging horizon for practical reasons. Choosing a contract that matures after the holding period avoids the roll-over risk and taking care of possible transaction costs that are not explicitly accounted in our equilibrium model.

is given by

$$\rho^{P,F}(\Delta,\tau) = \frac{B_P^\top \Omega(\Delta) B_F(\tau - \Delta)}{\sqrt{B_P^\top \Omega(\Delta) B_P} \sqrt{B_F(\tau - \Delta)^\top \Omega(\Delta) B_F(\tau - \Delta)}} \tag{25}$$

where $\Omega(\Delta)$ is the conditional covariance matrix at an horizon Δ for the state vector $L_t = \{\omega_t, i_t, p_t\}^\top$; and $B_P = \{0,0,1\}^\top$ and $B_F(t) = \{B_{F\omega}(t), B_{Fi}(t), 1\}^\top$ are vectors containing the weights of each state variable to form p_t and $\log(F(\omega_t, i_t, p_t, \tau))$, respectively.[10]

Proof See Appendix A.4. $\square$

Following the empirical literature, we define the correlation structure as the conditional correlation between inflation and futures returns on a contract that matures exactly at the same date as the hedging horizon, i.e., the correlation structure is $\rho^{P,F}(\Delta,\Delta)$.

Table 3 shows the matrix of correlations built upon equation (25) for $\Delta, \tau = 1, 2, 3$. The matrix of correlations is triangular superior because it is assumed that $\tau \geq \Delta$. The columns show, for a given maturity of the futures, the correlation between inflation and future returns across different holding periods, while the rows show the correlation for a given horizon for various futures contracts with different maturities. The main diagonal of the matrix corresponds to our definition of correlation structure.

Figure 1 presents the numerical values of the matrix of correlations using the estimates from Section 3 for holding periods up to 5 years and maturities $\tau = \Delta, \Delta + 1, \Delta + 2, \Delta + 3$. That is, the figure plots the diagonals of Table 3 from the main diagonal to the right, allowing to see the correlations for different holding period using futures that have at least the same maturity as the hedging horizon. The figure shows several interesting features of the matrix of correlations. First, the correlation values are consistent with those observed in the sample used, for example, the sample value for $\rho^{P,F}(1/12, 1/12)$ is 0.26. Second, for hedging horizons less than one year, the maximum possible correlation between inflation and future returns is achieved using futures with the same maturity as the holding period. This result is consistent with empirical studies such as Bodie and Rosansky (1980), Gorton and Rouwenhorst (2006) and Herbst (1985), that built indexes and analyzed hedging strategies satisfying the restriction $\tau = \Delta$ (although none of these papers discussed the possibility of hedging with contracts maturing after the holding period, i.e. $\tau \geq \Delta$). Third, for holding periods beyond one year, the differences between the correlations are almost indistinguishable. Therefore, in general, using oil futures with maturities $\tau = \Delta$ can be considered a useful rule of thumb. Fourth, as in Gorton and Rouwenhorst (2006), for all maturities the correlation between inflation and future returns increases with the holding period throughout the interval studied.

---

[10]Note that $\log(F(\omega_t, i_t, p_t, \tau)) = B_{F0}(\tau) + B_F(\tau)^\top L_t$.

It is important to note that the price level process in Proposition 2 follows a non-stationary process.[11] This implies that the state variables do not exhibit a covariance stationary process. However, as mentioned in the empirical section, the dynamics of the log ratio between the nominal futures and the price level (see equation (24)) is, indeed, stationary. This is the same as saying that the log futures and the log price level are cointegrated with cointegrating vector $(1, -1)$, implying that the unconditional correlation between inflation and futures returns converges to 1. Unfortunately, from a practical point of view this result is irrelevant, because the convergence is extremely slow. For example, if $\Delta = \tau = 10,000$ years, the correlation between the variables is slightly above 0.5.

The result stated in Proposition 4 is too general and does not allow for the full potential offered by the general equilibrium framework. Therefore, it becomes useful to review some special cases. We first consider the limiting case when the holding period tends to zero. This is equivalent to examine the instantaneous correlation between inflation and future returns for different maturities in the context of continuous trading or short-term hedging. This result is presented in the following corollary.

Corollary 1 *The instantaneous correlation between inflation and futures returns when the contract has a maturity of τ is*[12]

$$\lim_{\Delta \to 0} \rho^{P,F}(\Delta, \tau) = \frac{\sigma_{PK}^2 + \sigma_{PQ}^2 + \sigma_{PM}^2 + (\sigma_Q \sigma_{PQ} - \sigma_K \sigma_{PK}) B_{F\omega}(\tau)}{\sqrt{\sigma_{PK}^2 + \sigma_{PQ}^2 + \sigma_{PM}^2} \sqrt{(\sigma_{PK} - \sigma_K B_{F\omega}(\tau))^2 + (\sigma_{PQ} + \sigma_Q B_{F\omega}(\tau))^2 + \sigma_i^2 B_{Fi}^2(\tau) + \sigma_{PM}^2}} \quad (26)$$

This result shows that the instantaneous correlation between inflation and future returns depends only on the constant volatilities of the state variables (σ_K, σ_Q, σ_i and σ_M) and the deterministic functions of τ that go along with the state variables in the futures price equation. The covariance (numerator) is quite stable over time, because it is a linear function of $B_{F\omega}(\tau)$, which for our estimates has a very low variability. The standard deviation of the inflation, which is the first square root in the denominator, is constant. Most of the action comes from the standard deviation of the futures returns, which is the second square root in the denominator. The variance is quadratic on $B_{F\omega}(\tau)$ and $B_{Fi}(\tau)$ and increases initially with τ. This explains the decreasing pattern shown in Figure 1 for the different contracts when $\Delta = 0$.

Further analysis based on the economic grounds of the model can be done by considering the limit of (26) when τ tends to zero, therefore, the correlation no longer depends on the maturity τ. In this case the hedging instrument is the oil spot price itself. This result is presented in the following corollary.

---

[11]In more technical terms, the covariance stationarity of nominal returns depends on the eigenvalues of the matrix Ψ (see equation (A18) in Appendix 4). That is, the stationarity in the covariance requires that all eigenvalues are negative, a restriction that is violated, because in our model one of them is zero.

[12]This result can also be obtained using equations (19) and the dynamics of the futures prices, i.e.,

$$\lim_{\Delta \to 0} \rho^{P,F}(\Delta, \tau) = \frac{\mathbb{C}ov_t(dP_t/P_t, dF_t/F_t)}{\sqrt{\mathbb{V}ar_t(dP_t/P_t)}\sqrt{\mathbb{V}ar_t(dF_t/F_t)}}$$

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Corollary 2 *The instantaneous correlation between inflation and nominal oil spot returns is*

$$\lim_{\Delta,\tau\to0}\rho^{P,F}(\Delta,\tau)=\frac{\sigma_{PK}^2+\sigma_{PQ}^2+\sigma_{PM}^2-\alpha(\sigma_Q\sigma_{PQ}-\sigma_K\sigma_{PK})}{\sqrt{\sigma_{PK}^2+\sigma_{PQ}^2+\sigma_{PM}^2}\sqrt{(\sigma_{PK}+\alpha\sigma_K)^2+(\sigma_{PQ}-\alpha\sigma_Q)^2+\sigma_{PM}^2}} \tag{27}$$

To analyze this relationship, it is convenient to study how the price level and the nominal oil price are related. Equations (2), (15), (16) and (17) show that

$$P_t = \frac{M_t}{X_t}=c_{P1}\frac{M_t}{K_t^\alpha Q_t^{1-\alpha}} \tag{28}$$

$$S_t = P_tS_{rt}=\left(c_{P1}\frac{M_t}{K_t^\alpha Q_t^{1-\alpha}}\right)\left(c_{Sr1}\frac{K_t^\alpha}{Q_t^\alpha}\right)=c_{S1}\frac{M_t}{Q_t} \tag{29}$$

where c_{P1}, c_{Sr1} and c_{S1} are all positive constants. After applying Itô's lemma to equations (28)-(29) and using (7) we obtain

$$\frac{dP_t}{P_t} = \left(\frac{b_K-\alpha}{1-b_P}\right)\frac{dK_t}{K_t}-\left(\frac{1-\alpha}{1-b_P}\right)\frac{dQ_t}{Q_t}+\left(\frac{\sigma_M}{1-b_P}\right)dW_{Mt}+c_{P2}dt \tag{30}$$

$$\frac{dS_t}{S_t} = \left(\frac{b_K-\alpha}{1-b_P}+\alpha\right)\frac{dK_t}{K_t}-\left(\frac{1-\alpha}{1-b_P}+\alpha\right)\frac{dQ_t}{Q_t}+\left(\frac{\sigma_M}{1-b_P}\right)dW_{Mt}+c_{S2}dt \tag{31}$$

where c_{P2} and c_{S2} are constants. Equations (30)-(31) show that there are three independent components driving the covariance between the inflation and the oil spot return, since both processes share the same linear structure on $\frac{dK_t}{K_t}$, $\frac{dQ_t}{Q_t}$ and dW_{Mt}. We first study the effect of dW_{Mt}. An exogenous money supply shock affects simultaneously both returns, $\frac{dP_t}{P_t}$ and $\frac{dS_t}{S_t}$, by the same amount ($\frac{\sigma_M}{1-b_P}>0$). Therefore, at least from a theoretical point of view, this is an important driver of the positive correlation between the variables. This is a pure price level effect, because the nominal spot price is proportional to the price level. It should be noted that, since in our estimates $b_P<0$, the overall effect over $\frac{dP_t}{P_t}$ and $\frac{dS_t}{S_t}$ is partially reduced by the reaction of the monetary policy (see equation (7)), but without altering the correlation effect commented above. Using our estimates, 10.8% of the total covariance between inflation and oil returns is accounted by this component.

The second driver of the correlation is the effect produced by the shocks to the oil stocks. A negative shock due to changes in dW_{Qt}, would simultaneously increase inflation ($-\frac{1-\alpha}{1-b_P}<0$) and the nominal spot price return ($-(\frac{1-\alpha}{1-b_P}+\alpha)<0$), implying positive correlation between the variables. A sufficient restriction for having both variables reacting in the same direction is $b_P<0$, which makes sense and is satisfied in our estimates. Intuitively, a decrease in the oil stocks, increases the nominal spot price and also the price level (unless there is no oil in the consumption basket, i.e. $\alpha=1$). However, in this case the monetary policy also has an effect in the opposite direction, since $b_P<0$, an increase in the price level decreases the money supply M_t. This reduces the impact of a decrease in Q_t, but with only a minor effect on the correlation. These shocks are inflationary because of a sub-reaction of the monetary authority,

which reduces the money supply by a lower amount than necessary to prevent prices from rising. That is, this result is obtained because the authority does not explicitly consider changes in the supply of oil in its policy. Moreover, the impact is greater in $\frac{dS_t}{S_t}$ than in $\frac{dP_t}{P_t}$, i.e., an increase in oil prices tends to moderate the inflationary effect, because oil in not the only component on the consumption basket. For our estimates, this component has a minimum effect in the total covariance between inflation and oil returns (0.5%).

Finally, the last source of the correlation is related to the exogenous output shocks, dW_{Kt}. A positive shock to the capital stock would cause inflation ($\frac{b_K - \alpha}{1 - b_P} > 0$) and an increase in the nominal spot price ($\frac{b_K - \alpha}{1 - b_P} + \alpha > 0$) provided that $b_K > \alpha$ and $b_P < 0$, which is the case for our estimates. This source of positive correlation is also subject to the endogenous reaction of the monetary authority, similar to the one discussed in the previous paragraph. Importantly, these inflationary shocks are due to an over-reaction by the central bank ($b_K > \alpha$), which injected more money than necessary. Indeed, if the policy were to set $b_K = \alpha$, the output shocks would have no effect in the price level dynamics, reducing the volatility of this variable and this source of correlation with oil prices. On the other hand, if there were no endogenous monetary reaction to these shocks (i.e. $b_K = 0$), the output shocks would generate a negative correlation between $\frac{dP_t}{P_t}$ and $\frac{dS_t}{S_t}$ provided that $b_P < 0$. This could even offset positive correlation due to the two effects previously discussed. For our estimates, this last effect accounts for most of the covariance between inflation and oil returns (88.7%).

It is important to see that the standard argument used in the literature to justify a positive correlation between inflation and commodity prices is present in our model. Having $\alpha < 1$ implies that oil is in the consumption basket. However, as we have seen above, there are multiple mechanisms that may generate a positive correlation between inflation and oil returns. If we set $\alpha = 1$ and keep everything else equal, the instantaneous correlation drops to -0.54, basically influenced by the reaction of the monetary policy to the output shocks. This suggests the mechanisms discussed above play an important role in determining the positive correlation observed in the data.

Figure 2 shows how the correlation structure varies with the reaction of the monetary policy to the output shocks (b_K). The figure shows that the correlation is highly sensitive to the choice of the monetary policy parameter. For example, if the central bank's reaction to the shocks is the one needed to stabilize the price level ($b_K = \alpha$), the correlation between inflation and the nominal oil spot returns drops to values very close to zero. On the other hand, a greater sensitivity of the monetary policy to these type of shocks ($b_K = 2$), would rise the short-term correlation to almost one, but it would decrease with the holding period.

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5 Conclusions

In this paper we present a general equilibrium model of a monetary economy that is used to understand the economics behind the correlation between inflation and oil futures returns. A positive correlation between these variables has been documented in various empirical studies, which suggests that crude oil futures are good instruments to hedge inflation risk.

We derive a simple expression for the correlation between inflation and futures returns. The correlation has two temporal dimensions: the holding period or hedging horizon, and the maturity of the hedging instrument. We estimate the model using maximum likelihood and find that for holding periods below a year, the futures contracts that have a higher correlation with the inflation rate are those whose maturity date are closer to the hedging horizon. For longer holding periods, the difference of the correlations across maturities becomes insignificant. Our theoretical model verifies the findings of Gorton and Rouwenhorst (2006), who documents that the correlation is increasing with the holding period. We find that this result is valid for any futures with maturities longer than the hedging horizon.

When the holding period and maturity of the futures tend to zero, the drivers of the positive correlation between inflation and futures returns are the shocks to the money supply, to the oil stocks and output. Monetary shocks and oil stock shocks are natural sources of positive correlation. However, the sign of the correlation due to output shocks depends heavily on the reaction of the monetary authority to this type of shocks. For our estimates, we find that the central bank overreacts increasing the money supply, which basically rises the price level generating a positive correlation between inflation and futures returns. On the other hand, if the monetary policy were neutral to the output shocks this source of correlation would be negative. We also find that the effect of the inflation targeting policy has only a minor effect on the correlation between inflation and futures returns.

Among the many possible extensions of the model, we think it would be interesting to endogenize the oil investment decision of the agents. Also, the price of the producing firms could be obtained to study the correlation between crude oil and stock prices in a general equilibrium framework.

References

Bakshi, Gurdip S., and Zhiwu Chen, 1996, Inflation, asset prices, and the term structure of interest rates in monetary economies, *Review of Financial Studies* 9, 241–275.

Balduzzi, Pierluigi, 2007, Money and asset prices in a continuous-time Lucas and Stokey cash-in-advance economy, *Journal of Economic Dynamics and Control* 31, 2713 – 2743.

Bernanke, Ben S., Mark Gertler, and Mark W. Watson, 1997, Systematic monetary policy and the effects of oil price shocks, *Brookings Papers on Economic Activity* 1997, 91–157.

———— , 2004, Reply: Oil shocks and aggregate macroeconomic behavior: The role of monetary policy, *Journal of Money, Credit and Banking* 36, 287–291.

Bodie, Zvi, and Victor I. Rosansky, 1980, Risk and return in commodity futures, *Financial Analysts Journal* 36, 27–39.

Buraschi, Andrea, and Alexei Jiltsov, 2005, Inflation risk premia and the expectations hypothesis, *Journal of Financial Economics* 75, 429 – 490.

Carlson, Murray, Zeigham Khokher, and Sheridan Titman, 2007, Equilibrium exhaustible resource price dynamics, *Journal of Finance* 62, 1663–1703.

Casassus, Jaime, and Pierre Collin-Dufresne, 2005, Stochastic convenience yield implied from commodity futures and interest rates, *Journal of Finance* 60, 2283–2332.

———— , and Bryan Routledge, 2008, Equilibrium commodity prices with irreversible investment and non-linear technology, Working Paper, Columbia University.

Chen, Ren R., and Louis Scott, 1993, Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates, *Journal of Fixed Income* 3, 14–31.

Finn, Mary G., 2000, Perfect competition and the effects of energy price increases on economic activity, *Journal of Money, Credit and Banking* 32, 400–416.

Froot, Kenneth A ., 1995, Hedging portfolios with real assets, *Journal of Portfolio Management* 21, 60–77.

Gorton, Gary, and K. Geert Rouwenhorst, 2006, Facts and fantasies about commodity futures, *Financial Analysts Journal* 62, 47–68.

Greer, Robert J., 2000, The nature of commodity index returns, *Journal of Alternative Investments* 3, 4553.

Hamilton, James D., and Ana Maria Herrera, 2004, Comment: Oil shocks and aggregate macroeconomic behavior: The role of monetary policy, *Journal of Money, Credit and Banking* 36, 265–286.

Herbst, Anthony F., 1985, Hedging against price index inflation with futures contracts., *Journal of Futures Markets* 5, 489 – 504.

Janecek, Karel, and Steven E. Shreve, 2004, Asymptotic analysis for optimal investment and consumption with transaction costs, *Finance and Stochastics* 8, 181–206.

Kogan, Leonid, 2001, An equilibrium model of irreversible investment, *Journal of Financial Economics* 62, 201–245.

———— , Dmitry Livdan, and Amir Yaron, 2009, Oil futures prices in a production economy with investment constraints, *Journal of Finance* 64, 1345–1375.

Lioui, Abraham, and Patrice Poncet, 2005, General equilibrium pricing of CPI derivatives, *Journal of Banking and Finance* 29, 1265 – 1294.

Lucas, Robert E., and Nancy L. Stokey, 1987, Money and interest in a cash-in-advance economy, *Econometrica* 55, 491–513.

Pearson, Neil D., and Tong-Sheng Sun, 1994, Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model, *Journal of Finance* 49, 1279–1304.

Schwartz, Eduardo S., 1997, The stochastic behavior of commodity prices: Implications for valuation and hedging, *Journal of Finance* 52, 923–973.

Sidrauski, Miguel, 1967, Inflation and economic growth, *Journal of Political Economy* 75, 796–810.

Taylor, John B., 1993, Discretion versus policy rules in practice, *Carnegie-Rochester Conference Series on Public Policy* 39, 195 – 214.

Wei, Chao, 2003, Energy, the stock market, and the putty-clay investment model, *American Economic Review* 93, 311–323.

Appendix

A Proofs

This appendix contains the proofs of Propositions 1 to 4.

A.1 Proof of Proposition 1

To obtain the results in Proposition 1, first equation (14) is substituted into the HJB equation (12). Then, the optimal controls are obtained from the standard first order conditions with respect to C_t, Z_t, Y_t and γ_t. The solutions in terms of the partial derivatives of J are

$$C_t^* = \frac{\phi}{J_K}, \qquad Z_t^* = \frac{\alpha(1-\phi)}{J_K}, \qquad Y_t^* = \frac{(1-\phi)(1-\alpha)}{J_Q} \quad \text{and} \quad \gamma_t^* = \frac{A\eta K_t J_K}{J_Q} \tag{A1}$$

It only remains to obtain the value function, $J(K_t, Q_t, i_t)$, and replace it in the controls and in the real oil price. We replace the controls in equation (A1) in the HJB equation and guess a solution of the form:

$$J(K_t, Q_t, i_t) = \delta_0 + \left(\frac{1}{\beta} - \delta_\omega\right)\log(K_t) + \delta_\omega \log(Q_t) + \delta_i i_t \tag{A2}$$

This guess implies that the HJB equation is affine in $\omega_t \equiv \log\left(\frac{Q_t}{K_t}\right)$ and i_t, therefore, there is a perfectly identified system of three equations for δ_0, δ_ω and δ_i. The solution of this system is

$$
\begin{aligned}
\delta_0 &= \frac{1}{\beta}\left(\delta_i \kappa_i \theta_i + \left(\frac{1}{\beta} - \delta_\omega\right)\left(A\left(1 - \eta\left(1 - \log\left(A\eta\right)\right)\right) - \frac{\sigma_K^2}{2}\right) - \delta_\omega \frac{\sigma_Q^2}{2}\right. \\
&\qquad \left. - \left(1 - \log\left(\phi^\phi\left(\alpha^\alpha(1-\alpha)^{1-\alpha}(1-\phi)\right)^{1-\phi}\right)\right)\right) - \log\left(\left(\frac{1}{\beta} - \delta_\omega\right)^{\frac{1}{\beta} - \delta_\omega}\delta_\omega^{\delta_\omega}\right) \tag{A3}
\end{aligned}
$$

$$\delta_\omega = \frac{1}{\beta} - \frac{\alpha + \phi - \alpha\phi}{\beta + A\eta} \tag{A4}$$

$$\delta_i = \frac{1}{\beta + \kappa_i}\left(\delta_\omega(1 + \psi) - \frac{1}{\beta}\right) \tag{A5}$$

A.2 Proof of Proposition 2

To obtain the equilibrium price process, $\frac{dP_t}{P_t}$, we use the inter-temporal market clearing condition $\frac{dM_t^d}{M_t^d} = \frac{dM_t}{M_t}$. We guess an affine function form for the price process

$$
\begin{aligned}
\frac{dP_t}{P_t} &= \left(\mu_{P0} - \mu_{P\omega}\log(K_t) + \mu_{P\omega}\log(Q_t) + \mu_{Pi}i_t + \mu_{PM}\log(M_t)\right)dt \\
&\qquad + \sigma_{PK}dW_{Kt} + \sigma_{PQ}dW_{Qt} + \sigma_{Pi}dW_{it} + \sigma_{PM}dW_{Mt} \tag{A6}
\end{aligned}
$$

To obtain the parameters μ_{P0}, $\mu_{P\omega}$, μ_{Pi}, μ_{PM}, σ_{PK}, σ_{PQ}, σ_{Pi} and σ_{PM}, we replace equation (A6) in the money-demand and -supply processes and equate terms.

The resulting parameters are

$$
\mu_{P0} = \frac{1}{1-b_P}\left((b_K - \alpha)\left(A\left(1 - \eta\left(1 - \log\left(\frac{A\eta}{\delta_\omega}\left(\frac{1}{\beta} - \delta_\omega\right)\right)\right)\right)\right) - \beta - \frac{\alpha}{1-b_P}\sigma_K^2\right)
$$
$$
+ (1-\alpha)\left(\beta + \frac{1-\alpha}{1-b_P}\sigma_Q^2\right) + \mu_M - b_K\theta_K - b_P\theta_P + \alpha(1-\alpha)\left(\frac{\sigma_K^2}{2} + \frac{\sigma_Q^2}{2}\right)\right) \qquad \text{(A7)}
$$

$$
\mu_{P\omega} = \frac{A\eta(b_K - \alpha)}{1-b_P}, \qquad \mu_{Pi} = -\frac{\psi(1-\alpha) + b_K - \alpha}{1-b_P}, \qquad \mu_{PM} = 0 \qquad \text{(A8)}
$$

$$
\sigma_{PK} = \frac{\sigma_K(b_K - \alpha)}{1-b_P}, \qquad \sigma_{PQ} = -\frac{\sigma_Q(1-\alpha)}{1-b_P}, \qquad \sigma_{Pi} = 0, \qquad \sigma_{PM} = \frac{\sigma_M}{1-b_P} \qquad \text{(A9)}
$$

A.3 Proof of Proposition 3

The differential equation (21) satisfied by the futures is expressed under the risk-neutral measure $\mathbb{Q}$, thus, we need to obtain the nominal risk premiums associated with each one of the risk factors to make the proper change of measure. In this economy, a nominal flow x_t to be received at a future time $T > t$ is valued in the following way

$$
v_t^x = \mathbb{E}_t\left[\frac{\Gamma_T}{\Gamma_t}x_t\right] = \mathbb{E}_t\left[\frac{\Lambda_T}{\Lambda_t}\frac{P_t}{P_T}x_t\right] = \mathbb{E}_t\left[e^{-\beta(T-t)}\frac{U_X(X_T)}{U_X(X_t)}\frac{P_t}{P_T}x_t\right] \qquad \text{(A10)}
$$

where v_t^x is the present value of the flow x_t, $\Lambda_t = e^{-\beta t}\frac{U_X(X_t)}{U_X(X_0)}$ is the real pricing kernel, $U_X(X_t)$ is the marginal utility of the consumption basket and $\Gamma_t \equiv \frac{\Lambda_t}{P_t}$ is the nominal pricing kernel. Using (1) and (2) the nominal pricing kernel can be written as follows

$$
\Gamma_t = \frac{\Lambda_t}{P_t} = \frac{e^{-\beta t}U_X(X_t)}{P_t U_X(X_0)} = \frac{e^{-\beta t}X_0}{P_t X_t} \qquad \text{(A11)}
$$

Applying Itô's lemma to (A11) to obtain the dynamic of Γ_t results in

$$
\frac{d\Gamma_t}{\Gamma_t} = -r_{nt}dt - \lambda_K dW_{Kt} - \lambda_Q dW_{Qt} - \lambda_i dW_{it} - \lambda_M dW_{Mt} \qquad \text{(A12)}
$$

where r_{nt} is the nominal interest and the λ_j's are the following market prices of risk

$$
\lambda_K = \alpha\sigma_K + \sigma_{PK}, \qquad \lambda_Q = (1-\alpha)\sigma_Q + \sigma_{PQ}, \qquad \lambda_i = 0, \quad \text{and} \quad \lambda_M = \sigma_{PM} \qquad \text{(A13)}
$$

To change from the physical measure to the risk-neutral measure Q, the following substitutions must be made

$$
dW_{jt} = dW_{jt}^{\mathbb{Q}} - \lambda_j dt \qquad \text{(A14)}
$$

To obtain the solution of the futures price we find $F(\omega_t, i_t, p_t, \tau)$ such that the risk-neutral valuation equation (21) is satisfied. First, we apply Itô's lemma to the guess function (23) in Proposition 3. We change the processes from the physical to the risk-neutral measure using equation (A14), and take the expected value $\mathbb{E}_t[F_t]$. This resulting equation is affine on ω_t and i_t, and must be equal to zero for any $t \le T$. As it's well know form the affine asset pricing literature, this equation implies a system of ODE's for $B_{F0}(\tau)$, $B_{F\omega}(\tau)$ and $B_{Fi}(\tau)$. From the boundary condition in (22), we obtain the initial values for $B_{F0}(0)$, $B_{F\omega}(0)$ and $B_{Fi}(0)$. Solving the system of ODE's yields

$$
B_{F\omega}(\tau) = (1 - e^{-A\eta\tau})\left(\alpha + \frac{\mu_{P\omega}}{A\eta}\right) - \alpha \qquad \text{(A15)}
$$

20

$$B_{Fi}(\tau) \;=\; \frac{1 - e^{-\kappa_i \tau}}{\kappa_i}\left(\mu_{Pi} + \frac{(1+\psi)\mu_{P\omega}}{A\eta}\right) \tag{A16}$$

We skip the solution of $B_{F0}(\tau)$ because it's messy and uninformative.

A.4 Proof of Proposition 4

The correlation between two variables is obtained as the quotient between its covariance and the product of their standard deviations. This appendix explains the calculation of each one of these components. The conditional covariance between times t and $t+\Delta$ of the inflation rate $\pi(t+\Delta) = p_{t+\Delta} - p_t$ and the oil futures returns with maturity τ, $R(t+\Delta) = \log(F(L_{t+\Delta}, \tau-\Delta)) - \log(F(L_t, \tau))$, is given by

$$
\begin{aligned}
\varphi^{P,F}(\Delta,\tau) &= \mathbb{C}ov_t(\pi(t+\Delta), R(t+\Delta)) \\
&= \mathbb{E}_t\left[(p_{t+\Delta} - \mathbb{E}_t[p_{t+\Delta}])(\log(F(L_{t+\Delta},\tau-\Delta)) - \mathbb{E}_t[\log(F(L_{t+\Delta},\tau-\Delta))])^\top\right] \\
&= \mathbb{E}_t\left[B_P^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}])(B_F(\tau-\Delta)^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}]))^\top\right] \\
&= B_P^\top\Omega(\Delta)B_F(\tau-\Delta)
\end{aligned}
\tag{A17}
$$

where $\tau \geq \Delta$ and $\Omega(\Delta)$, the covariance matrix of the state vector $L_{t+\Delta}$ conditional on L_t is

$$\Omega(\Delta) \equiv \mathbb{E}_t\left[(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}])(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}])^\top\right] = \int_0^\Delta e^{\Psi(\Delta-u)}\Sigma e^{\Psi^\top(\Delta-u)}du \tag{A18}$$

with Ψ and Σ satisfying $\mathbb{E}_t(dL_t) = (U + \Psi L_t)dt$ and $\mathbb{V}ar_t(dL_t) = \Sigma dt$. On the other hand, the conditional variance between t and $t+\Delta$ of the inflation rate is

$$
\begin{aligned}
\varphi^{P,P}(\Delta,\tau) &= \mathbb{V}ar_t(\pi(t+\Delta)) \\
&= \mathbb{E}_t\left[(p_{t+\Delta} - \mathbb{E}_t[p_{t+\Delta}])(p_{t+\Delta} - \mathbb{E}_t[p_{t+\Delta}])^\top\right] \\
&= \mathbb{E}_t\left[B_P^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}])(B_P^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}]))^\top\right] \\
&= B_P^\top\Omega(\Delta)B_P
\end{aligned}
\tag{A19}
$$

Finally, the conditional variance between t and $t+\Delta$ of the futures returns with maturity τ, $R(t+\Delta)$, is

$$
\begin{aligned}
\varphi^{F,F}(\Delta,\tau) &= \mathbb{V}ar_t(R(t+\Delta)) \\
&= \mathbb{E}_t[(\log(F(L_{t+\Delta},\tau-\Delta)) - \mathbb{E}_t[\log(F(L_{t+\Delta},\tau-\Delta))]) \\
&\qquad \cdot(\log(F(L_{t+\Delta},\tau-\Delta)) - \mathbb{E}_t[\log(F(L_{t+\Delta},\tau-\Delta))])^\top] \\
&= \mathbb{E}_t\left[B_F(\tau-\Delta)^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}])(B_F(\tau-\Delta)^\top(L_{t+\Delta} - \mathbb{E}_t[L_{t+\Delta}]))^\top\right] \\
&= B_F(\tau-\Delta)^\top\Omega(\Delta)B_F(\tau-\Delta)
\end{aligned}
\tag{A20}
$$

Then, Proposition (4) results from combining (A17), (A19) and (A20) to build the conditional correlation between the inflation rate and oil futures returns.

Table 1: **Historical and model implied moments**

The table shows the historical and the model implied moments and errors for the time-series data.

	Historical data		Model			
	Sample average	Sample stdev.	Estimate average	Estimate stdev.	MSE	MAE (n=210)
1-month futures prices (US$/bbl)	37.53	25.61	38.10	26.35	3.05	1.24
3-months futures prices (US$/bbl)	37.62	26.00	37.83	26.33	0.50	0.51
6-months futures prices (US$/bbl)	37.49	26.33	37.49	26.33	0.00	0.00
9-months futures prices (US$/bbl)	37.31	26.53	37.21	26.35	0.12	0.25
12-months futures prices (US$/bbl)	37.13	26.65	37.00	26.39	0.26	0.36
18-months futures prices (US$/bbl)	36.85	26.75	36.72	26.48	0.28	0.36
24-months futures prices (US$/bbl)	36.67	26.75	36.59	26.60	0.10	0.22
30-months futures prices (US$/bbl)	36.57	26.73	36.57	26.73	0.00	0.00
Nominal 3-months yield (annual)	0.036	0.018	0.012	0.003	0.00	0.02
Expected inflation (annual)	0.026	0.034	0.024	0.009	0.00	0.02
Expected money growth rate (annual)	0.054	0.024	0.060	0.003	0.00	0.02

Table 2: **Parameter estimates**

Maximum Likelihood estimates for the sample periods Jul-1992 to Dec-2009. The parameters marked with an asterisk (*) were estimated, while the others were calibrated.

Parameter		Estimate	Standard error	t-stat
β		0.050		
ϕ		0.687		
α		0.912		
η		0.040		
σ_K	(*)	0.333	0.017	19.55
A	(*)	0.143	0.020	6.99
ψ		52.30		
σ_Q		0.015		
κ_i	(*)	0.649	0.041	15.93
θ_i	(*)	0.001	0.000	1.30
σ_i	(*)	0.003	0.000	13.78
μ_M	(*)	0.056	0.013	4.26
b_K	(*)	0.946	0.189	5.01
θ_K		0.037		
b_P	(*)	-0.391	0.098	-4.00
θ_P		0.026		
σ_M		0.024		
ρ_1	(*)	0.535	0.027	19.68
ρ_2	(*)	0.994	0.006	177.60
ρ_3	(*)	0.483	0.062	7.84
ρ_4	(*)	0.962	0.015	65.36
Log-likelihood		8019.1026		

Table 3: **Conditional correlation**

Conditional correlation matrix between inflation and oil futures returns considering a holding period Δ and contracts with different maturity τ.

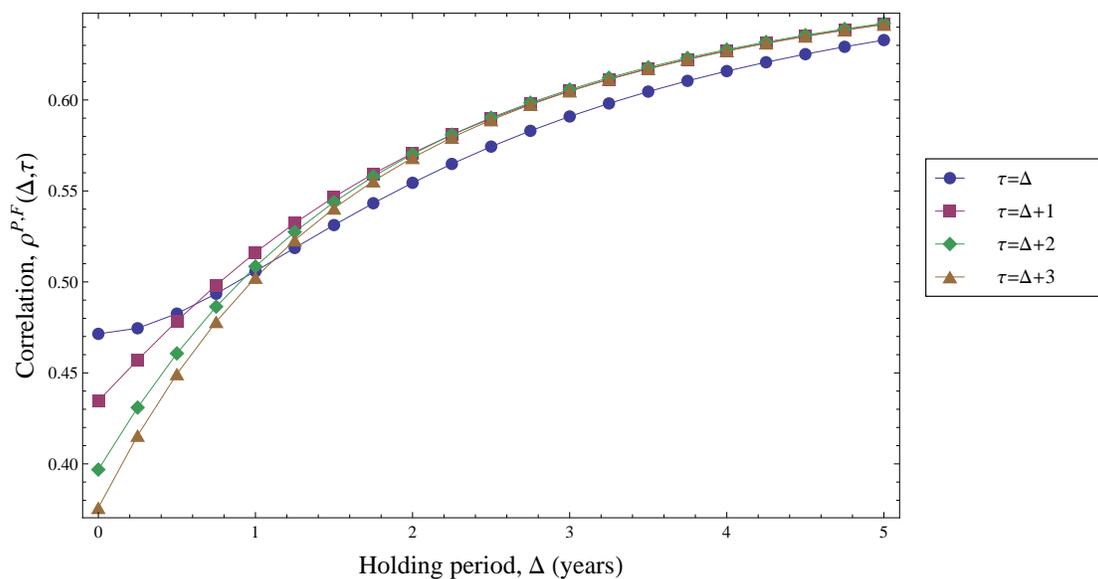|	$\tau = 1$	$\tau = 2$	$\tau = 3$
$\Delta = 1$	$\rho^{P,F}(1,1)$	$\rho^{P,F}(1,2)$	$\rho^{P,F}(1,3)$
$\Delta = 2$		$\rho^{P,F}(2,2)$	$\rho^{P,F}(2,3)$
$\Delta = 3$			$\rho^{P,F}(3,3)$

Figure 1: Diagonals of the matrix of conditional correlations in Table 3.
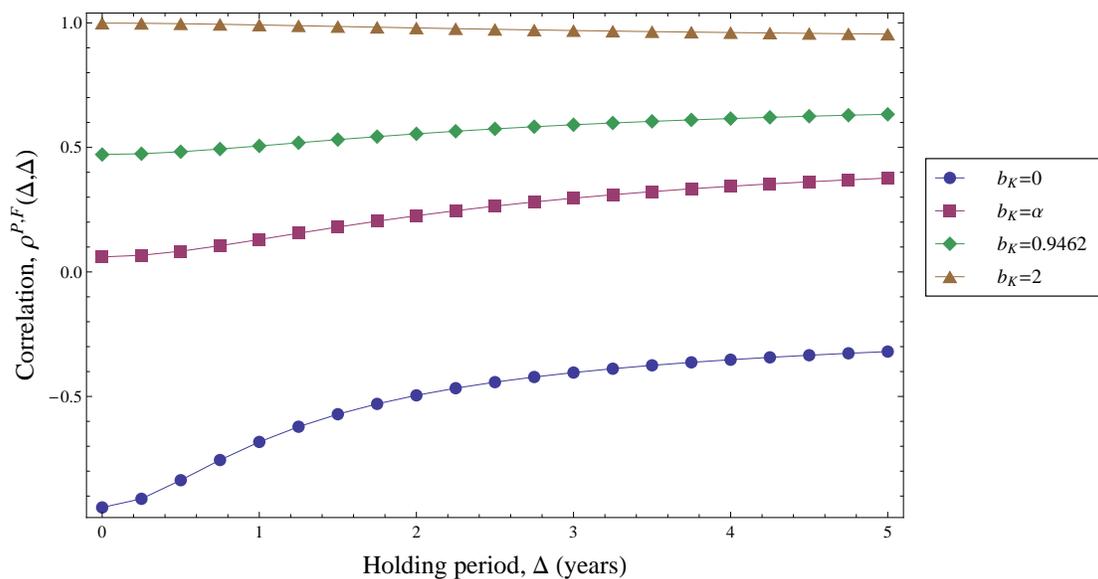

Figure 2: Effect of the monetary policy reaction to the output shocks in the correlation structure.